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LETTER TO THE EDITOR

Modulational instabilities and soliton solutions of a generalized nonlinear Schrödinger equation

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Abstract. Previous forms of the generalized nonlinear Schrödinger equation are ambiguously related to the corresponding nonlinear dispersion relation. In the present paper we therefore propose a Hamiltonian equation which is determined in a unique way. Soliton solutions of the nonlinear Schrödinger equations are found, and modulational instabilities are investigated.

The well known cubic nonlinear Schrödinger equation (NSE) [1, 2] has some remarkable integrability properties [3]. However, in many applications it contains also some small additional terms which destroy these properties. The equation is then generally studied by means of perturbation techniques [4]. An interesting example [5] is, in normalized variables, the equation

$$iu_t + u_{xx} + 2p|u|^2u = -C|u_x|^2/u^* \tag{1}$$

where $u = u(x, t)$ represents the wave amplitude, $u_t \equiv \partial u/\partial t$, $u_x \equiv \partial u/\partial x$, $p = +1$ or -1, C is a constant, and the asterisk denotes the complex conjugate. When $C = 0$, we note that the two values (± 1) of p correspond to the two different versions of the usual unperturbed NSE [3]. Equation (1), which is not Hamiltonian, conserves the wave action

$$N = \int_{-\infty}^{\infty} dx|u|^2. \tag{2}$$

A related equation

$$iu_t + u_{xx} + 2p|u|^2u = -Cu_x^2/u \tag{3}$$

was noticed previously [4, 6]. This equation has, however, no conservation laws. It describes the propagation of nonlinear surface waves on a plasma with a sharp boundary [7]. In [5, 6] it was stressed that the usual procedure [1, 2, 3] that associates the NSE with the weak-dispersion, weak-nonlinearity expansion of the nonlinear dispersion relation

$$\Omega = \Omega(K, |u|^2) \tag{4}$$

is ambiguous, and that (1) and (3), as well as the usual NSE (where $C = 0$), agree with (4).

It is well known that the usual NSE can be derived from the Hamiltonian density [3]

$$\mathcal{H}_0 = |u_x|^2 - p|u|^4. \tag{5}$$

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Equations (1) and (3) are not Hamiltonian, however. We therefore have to find a Hamiltonian equation that is similar to these equations. Guided by previous work [1-6] we thus look for the simplest additional term $\mathcal{H}_1$ to the Hamiltonian, which is consistent with the nonlinear dispersion relation (4) as well as with the basic part $\mathcal{H}_0$. It is easy to verify that this term is

$$\mathcal{H}_1 = C(u/u^*)(u_x^*)^2 + C^*(u^*/u)u_x^2 \tag{6}$$

where C is an arbitrary complex parameter. The equation of motion that corresponds to the full Hamiltonian density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ is then

$$\mathrm{i}u_t + u_{xx} + 2p|u|^2u = C^*u_x^2/u + C(u_x^*)^2u/(u^*)^2 - 2Cu_{xx}^*u/u^* - 2C|u_x|^2/u^*. \tag{7}$$

Equation (7) conserves the wave action (2) as well as the energy

$$H = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathcal{H} \tag{8a}$$

and the momentum

$$P = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, u_x^*u. \tag{8b}$$

Furthermore we note that (7), and also (3), is invariant with respect to Galilei transformations, whereas (1) does not have this property.

We shall now analyse the continuous-wave (CW) and soliton solutions of equations (1), (3) and (7). We then look for soliton solutions of the form

$$u_{\mathrm{sol}} = 2\mathrm{i}A\eta[\mathrm{sech}(2\eta x)]\exp(4\mathrm{i}\omega\eta^2 t) \tag{9}$$

where the inverse soliton width η is an arbitrary parameter, and where A and ω are constants that have to be specified. Inserting first (9) into (1) or (3) one thus finds

$$\omega = 1 + C \tag{10a}$$

and

$$pA^2 = 1 + C/2. \tag{10b}$$

We note that $\omega = pA^2 = 1$ for the usual NSE [1-4].

The soliton (9) is immobile. By means of a Galilei transformation one directly finds the corresponding soliton solution of equation (3) that moves with an arbitrary velocity V. Thus

$$u_{\mathrm{sol}} = 2\mathrm{i}A\eta[\mathrm{sech}(2\eta(x - Vt))]\exp[\mathrm{i}Vx/2(1+C) + 4\mathrm{i}\omega\eta^2 t - \mathrm{i}V^2t/4(1+C)]. \tag{11a}$$

The moving soliton solution of equation (1) that is not Galilei invariant, can be found by means of some algebra. It is

$$u_{\mathrm{sol}} = 2\mathrm{i}A\eta[\mathrm{sech}(2\eta(x - Vt))]\exp[\mathrm{i}Vx/2 + 4\mathrm{i}\omega\eta^2 t - \mathrm{i}(1-C)V^2t/4]. \tag{11b}$$

We note that the soliton solutions (11*a*) and (11*b*) are different, unless V is zero. Furthermore, it is evident from (10*b*) that soliton solutions exist if

$$p(1 + C/2) > 0. \tag{12}$$

The inequality (12) agrees of course with the existence condition $p = +1$ for the usual NSE, where $C = 0$. The CW solutions of (1) and (3) are written in the standard form

$$u_{\mathrm{CW}} = a_0\exp(\mathrm{i}kx - \mathrm{i}\omega_0 t). \tag{13}$$

Inserting (13) into equation (1) one finds

$$\omega_0 = (1-C)k^2 - 2pa_0^2 \tag{14}$$

whereas (13) into (3) yields

$$\omega_0 = (1+C)k^2 - 2pa_0^2. \tag{15}$$

In order to study the modulational instability [8] of the CW solutions, it is convenient to express u in terms of its real amplitude $a(x, t)$ and phase $\phi(x, t)$, i.e.

$$u(x, t) = a(x, t) \exp[\mathrm{i}\phi(x, t)]. \tag{16}$$

Equation (1) is thus equivalent to the system

$$a_t + a\phi_{xx} + 2a_x\phi_x = 0 \tag{17a}$$

and

$$-a\phi_t + a_{xx} - a\phi_x^2 + 2pa^3 = -C(a_x^2 + a^2\phi_x^2)/a \tag{17b}$$

and the CW solution (13) corresponds to $a = a_0$, $\phi = \phi_0 = kx - \omega_0 t$.

The perturbed CW solution is now written in the form

$$a = a_0 + a_1 \exp(\gamma t + \mathrm{i}qx) \tag{18a}$$

and

$$\phi = \phi_0 + \phi_1 \exp(\gamma t + \mathrm{i}qx) \tag{18b}$$

where a_1 and ϕ_1 represent small amplitude and phase perturbations with wavenumber q and instability growth rate γ. Linearizing equations (17) one thus obtains the dispersion relation that relates γ to q, or

$$\gamma_1^2 - 2\mathrm{i}Ckq\gamma_1 + q^2(q^2 - 4pa_0^2) = 0 \tag{19a}$$

where

$$\gamma_1 = \gamma + 2\mathrm{i}kq. \tag{19b}$$

It is easy to see that the condition for modulational stability, $\mathrm{Re}[\gamma(q)] \leqslant 0$ for all real q, corresponds to the inequality

$$k^2 \geqslant 4pa_0^2/C^2. \tag{20}$$

Similarly, we write (3) in terms of a and ϕ. Thus

$$a_t + a\phi_{xx} + 2a_x\phi_x = -2Ca_x\phi_x \tag{21a}$$

and

$$-a\phi_t + a_{xx} - a\phi_x^2 + 2pa^3 = -C(a_x^2 - a^2\phi_x^2)/a. \tag{21b}$$

Considering the perturbed solution (18), one obtains now, instead of (19), the equation

$$\gamma_2^2 + q^2(q^2 - 4pa_0^2) = 0 \tag{22a}$$

where

$$\gamma_2 = \gamma + 2\mathrm{i}kq(1+C). \tag{22b}$$

From (22) we obtain the usual Benjamin–Feir modulational stability condition [8]

$$pa_0^2 \leqslant 0. \tag{23}$$

By comparing the CW stability criteria (20) and (23) with the inequality (12) that defines the existence of the soliton, we conclude that, using the generalized non-Hamiltonian equations (1) and (3), a soliton may co-exist with a stable CW. This is not the case for the usual NSE. The wavenumber dependence of (20) is related to the Galilei-non-invariant character of (1).

Let us now turn our attention to the generalized Hamiltonian equation (7). A straightforward analysis shows then that a soliton solution of the form (9) can exist only if C is real and satisfies the inequality (cf (12))

$$p(1+2C)\geqslant 0. \tag{24}$$

The coefficients ω and A are (cf equations (10))

$$\omega = 1-2C \tag{25a}$$

and

$$pA^2 = 1+2C. \tag{25b}$$

A moving soliton (cf equations (11)) can be found by means of a Galilei transformation. Thus

$$u = 2\mathrm{i}A\eta[\operatorname{sech}(2\eta(x-Vt))]\exp[\mathrm{i}Vx/2(1-2C)+4\mathrm{i}\omega\eta^2 t-\mathrm{i}V^2 t/4(1-2C)]. \tag{26}$$

In the general case, when the parameter C is complex, the Galilei transformation of an arbitrary solution of (7) looks like (26) with C substituted by Re C.

The CW solution of (7) is given by (13) with (cf (14) and (15))

$$\omega_0 = (1-2\,\mathrm{Re}\,C)k^2 - 2pa_0^2. \tag{27}$$

We note that the CW solution, unlike the soliton, exists for arbitrary complex C.

The equations for the amplitude and phase that correspond to (7) are (cf (17) and (21))

$$a_t + a\phi_{xx} + 2a_x\phi_x = 2\,\mathrm{Re}\,C(a\phi_{xx}+2a_x\phi_x) - 2\,\mathrm{Im}\,C(a_{xx}+a_x^2/a) \tag{28a}$$

and

$$-a\phi_t + a_{xx} - a\phi_x^2 + 2pa^3 = -2\,\mathrm{Re}\,C(a\phi_x^2 + a_{xx}) - 2\,\mathrm{Im}\,Ca\phi_{xx}. \tag{28b}$$

After some algebra one obtains the dispersion relation that determines the modulational stability of the CW solution (cf (19) and (22))

$$\gamma_3^2 + q^4(1-4|C|^2) - 4q^2pa_0^2(1-2\,\mathrm{Re}\,C) = 0 \tag{29a}$$

where

$$\gamma_3 = \gamma + 2\mathrm{i}kq(1-2\,\mathrm{Re}\,C). \tag{29b}$$

The modulational stability condition is thus

$$|C|\leqslant 1/2 \tag{30a}$$

and

$$p<0. \tag{30b}$$

The inequality (30b) tells us that, as for the usual NSE, the CW solution of equation (7) is stable only if $p=-1$. Comparing (30a) and (30b) with (24) we notice that the soliton cannot exist if the CW is stable. This conclusion is the same as that for the usual NSE. There is also a parameter region $p=-1$, $C>\frac{1}{2}$, with real C, where the CW is unstable and there are no solitons.

Finally we note that in the particular case $p=0$ and $C=-\frac{1}{2}$, equation (7) is

$$\mathrm{i}u_t+u_{xx}=-u_x^2/2u-(u_x^*)^2u/2(u^*)^2+u_{xx}^*u/u^*+|u_x|^2/u^*. \qquad (31)$$

Although (31) contains no cubic term it admits soliton solutions of the form (9) with arbitrary A and with $\omega=2$ (cf (25)). The CW solutions of (31) have the frequency $\omega_0=2k^2$ (cf (27)) and they are modulationally stable according to (29).

An interesting problem which remains to be solved concerns the stability of the soliton solutions considered above. If the solitons are stable, it is also worthwhile to simulate collisions between them numerically, in order to see how inelastic the collisions are. In addition, when the CW solutions are stable one should look for modulated periodic solutions (cnoidal waves). Two-dimensional versions of the generalized NSE may also be of some interest. Thus one should investigate how the well known weak collapse of the usual two-dimensional NSE is influenced by our additional terms.

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